TOPICS IN COMPLEX ANALYSIS @ EPFL, FALL 2024
SOLUTION SKETCHES TO HOMEWORK 6

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Homework 6.1 (Holomorphic interpolation*)**.** Let $(a_n)_{n\in\mathbf{N}}$ be sequence without an accumulation point and let $(b_n)_{n\in\mathbf{N}}$ be an arbitrary sequence (both in $\mathbf{C}$). Show there exists a holomorphic function $f\colon \mathbf{C}\to\mathbf{C}$ such that $f(a_n) = b_n$ for every $n \in \mathbf{N}$[1].

Homework 6.2 (Representation of meromorphic functions as quotients)**.** Let $h\colon \mathbf{C}\to\mathbf{C}$ be a meromorphic function, i.e. its singularities are isolated and only poles of finite order. Show that there exist two entire functions $f, g\colon \mathbf{C}\to\mathbf{C}$ with no common zeros with $h = f/g$.

Solution. Without loss of generality we may and will assume h does not vanish identically. Let us denote by $P(h)$ the set of poles of h. Then $P(h)$ and $Z(h)$ are disjoint discrete sets. Due to the Weierstraß product theorem we find an entire function $g\colon \mathbf{C}\to\mathbf{C}$ such that $Z(g) = P(h)$ and $o_z(g)$ is equal to the order of the pole of h in $z \in P(h)$. Then g does not vanish identically. Setting $f = g\,h$ it follows f can be extended to an entire function since it is bounded near each point $z \in P(g)$. By construction we have $Z(f) \cap Z(g) = \emptyset$.

Homework 6.3 (Weierstraß product theorem on open sets)**.** Let $U \subset \mathbf{C}$ be an open set and let $(a_n)_{n\in\mathbf{N}}$ be a discrete sequence in U. Set $o_n := \#\{k \in \mathbf{N} : a_k = a_n\}$ and assume $o_n < \infty$ for every $n \in \mathbf{N}$. We claim there exists a holomorphic function $f\colon U \to \mathbf{C}$ such that $Z(f) = \{a_n : n \in \mathbf{N}\}$ and $o_{a_n}(f) = o_n$ for all $n \in \mathbf{N}$. Moreover, as we shall see in the proof the function f can be taken as an infinite product. The argument splits into two steps. Similarly to the Mittag–Leffler theorem we recenter part of the Weierstraß factors and replace $E_n(z/a_n)$ by $E_n((a_n - c_n)/(z - c_n))$ for a suitable sequence $(c_n)_{n\in\mathbf{N}}$. We denote by $S' \subset \mathbf{C}$ the set of accumulation points of the sequence $(a_n)_{n\in\mathbf{N}}$. If $S' = \emptyset$ there is nothing to prove as we can apply Theorem 4.2 from the lecture notes. Hence assume that $S' \neq \emptyset$.

a. Suppose there exists a sequence $(c_n)_{n\in\mathbf{N}}$ in S' such that $|a_n - c_n| \to 0$ as $n \to \infty$. Show the infinite product
$$f(z) := \prod_{n=1}^{\infty} E_n\Big[\frac{a_n - c_n}{z - c_n}\Big]$$
converges locally normally on U and obeys $Z(f) = \{a_n : n \in \mathbf{N}\}$ and $o_{a_n}(f) = o_n$ for every $n \in \mathbf{N}$.

b. Split the set $S := \{a_n : n \in \mathbf{N}\}$ as in Lemma 2.7 from the lecture notes and use Lemma 2.8 to conclude the proof by combining a. and Theorem 4.2.

Solution. a. Let $K \subset U$ be compact. Then there is $\varepsilon > 0$ such that $\operatorname{dist}(K, \partial U) \geq \varepsilon$. Since $c_n \in S' \subset \partial U$ we deduce there exists $n(K) \in \mathbf{N}$ such that for every $n \geq n(K)$,
$$K \subset \{z \in U : |z - c_n| \geq 2\,|a_n - c_n|\}.$$
Indeed, this holds because $|z - c_n| \geq \varepsilon$ for every $z \in K$ yet $|a_n - c_n| \to 0$ as $n \to \infty$. Hence, for every $n \geq n(K)$ we habe $|a_n - c_n|/|z - c_n| \leq 1/2$, so that Lemma 4.1 implies
$$\sum_{n\geq n(K)} \sup_{z\in K} \Big| E_n\Big[\frac{a_n - c_n}{z - c_n}\Big] - 1\Big| \leq \sum_{n \geq n(K)} 2^{-(n+1)} < \infty.$$

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[1]**Hint.** Combine the Weierstraß product theorem with the Mittag–Leffler theorem for a suitable sequence of principal parts.

Thus the infinite product f converges locally normally on U. Since each E_n has a simple zero in $z = 1$ and $a_n \neq c_n$ for every $n \in \mathbf{N}$ (recall $a_n \in U$ and $c_n \notin U$) we deduce from Lemma 3.11 that $Z(f) = \{a_n : n \in \mathbf{N}\}$ and $o_{a_n}(f) = o_n$ for every $n \in \mathbf{N}$.

b) Splitting $S = S_1 \sqcup S_2$ as in Lemma 2.7 we obtain $S_1 = \{a_{n,1} : n \in \mathbf{N}\}$ is closed and therefore $S' = S_2'$. Due to Lemma 2.8, the set $S_2 = \{a_{n,2} : n \in \mathbf{N}\}$ is discrete and such that there exists a sequence $(c_n)_{n\in\mathbf{N}}$ in S_2' such that $|c_n - a_{n,2}| \to 0$ as $n \to \infty$. Since S_1 is closed it is also discrete in $\mathbf{C}$ and therefore we can apply Theorem 4.2 to obtain that the entire function $f_1 \colon \mathbf{C} \to \mathbf{C}$ defined by

$$f_1(z) = \prod_{n=1}^{\infty} E_n\left[\frac{z}{a_{n,1}}\right]$$

has its only zeros with the correct multiplicity in $\{a_{n,1} : n \in \mathbf{N}\}$ (here we also used $0 \notin S_1$).

On S_2 we can apply a. to deduce the holomorphic function $f_2 \colon U \to \mathbf{C}$ given by

$$f_2(z) = \prod_{n=1}^{\infty} E_n\left[\frac{a_{n,2} - c_n}{z - c_n}\right]$$

has its only zeros with the correct multiplicity in $\{a_{n,2} : n \in \mathbf{N}\}$. Since S_1 and S_2 are disjoint the product $f = f_1 f_2$ satisfies the desired properties.

Homework 6.4 (First applications of Picard's little theorem)**.** a. Show every meromorphic function f that omits three distinct values $a, b, c \in \mathbf{C}$ is constant.

b. Give an example of a meromorphic function that omits the two values $0, 1 \in \mathbf{C}$.

c. Let $f \colon \mathbf{C} \to \mathbf{C}$ be an entire function. Show $f \circ f$ has a fixed point unless f is affine but not linear, i.e. $f(z) = z + b$ for some $b \in \mathbf{C} \setminus \{0\}$[2].

Solution. a. Assume f is nonconstant. The assignment $(f(z) - a)^{-1}$ can be extended to a nonconstant, entire function which omits the two different values $(b - a)^{-1}$ and $(c - a)^{-1}$. This contradicts Picard's little theorem.

b. Consider the meromorphic assignment $f(z) := (1 + \mathrm{e}^z)^{-1}$, which has first order poles in all points of the form $(2n + 1)\pi\mathrm{i}$, where $n \in \mathbf{Z}$.

c. Suppose $f \circ f$ has no fixed points. Then the assignment

$$g(z) = \frac{f(f(z)) - z}{f(z) - z}$$

is entire and $g(z) \neq 0$ for every $z \in \mathbf{C}$.

Moreover, note $g(z) \neq 1$ for every $z \in \mathbf{C}$. Indeed, $g(z) = 1$ implies $f(z)$ is a fixed point for f and therefore also for $f \circ f$. Hence by Picard's little theorem, $c := g(z) \in \mathbf{C} \setminus \{0, 1\}$. Differentiation yields

$$f'(z)\big[f'(f(z)) - c\big] = 1 - c.$$

Since $c \neq 1$ we know $f'(z) \neq 0$ and $f'(f(z)) \neq c$ for every $z \in \mathbf{C}$. Thus $f' \circ f$ is entire and omits the values 0 and $c \neq 0$. Picard's little theorem implies it is constant. Combining the open mapping theorem (for f) and the identity theorem (for f') it follows f' is constant. Hence f is of the form $f(z) = a\,z + b$ for every $z \in \mathbf{C}$. Since also f has no fixed point we conclude $a = 1$ and $b \neq 0$, as claimed.

[2]**Hint.** Consider the assignment

$$g(z) := \frac{f(f(z)) - z}{f(z) - z}$$

Show it is constant and differentiate it. Then deduce $f' \circ f$ omits the value 0 and said constant. Finally, show this implies that f' is constant.